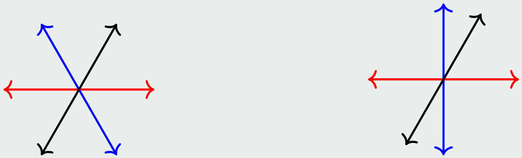
Saturating the Welch Bound for Frames over Finite Fields

Ian Jorquera
Joint with: Emily J King

Colorado State University
College of Natural Sciences,
Mathematics Department

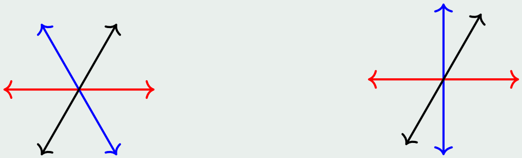
May 21, 2025

Line Packings: Can you pack n lines in $\mathbb{R}^d$ or $\mathbb{C}^d$, where every line is maximally spread apart?



Goal: Maximize pairwise acute angles, or minimize $\cos^2\theta$

Line Packings: Can you pack n lines in $\mathbb{R}^d$ or $\mathbb{C}^d$, where every line is maximally spread apart?



Goal: Maximize pairwise acute angles, or minimize $\cos^2\theta$

Given n lines, represent each by a unit vector

$$\Phi = \begin{bmatrix} | & | & & | \\ \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ | & | & & | \end{bmatrix} \in \mathbb{F}^{d\times n}$$

New Goal: Minimize $\max_{i\neq j} |\langle \varphi_i, \varphi_j \rangle|^2$

Finding Optimal Packings

Given n lines in $\mathbb{R}^d$ or $\mathbb{C}^d$, represented each by a unit vector

$$\Phi = \begin{bmatrix} | & | & & | \\ \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ | & | & & | \end{bmatrix}$$

New Goal: Minimize $\max_{i \neq j} |\langle \varphi_i, \varphi_j \rangle|^2$

Finding Optimal Packings

Given n lines in $\mathbb{R}^d$ or $\mathbb{C}^d$, represented each by a unit vector

$$\Phi = \begin{bmatrix} | & | & & | \\ \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ | & | & & | \end{bmatrix}$$

New Goal: Minimize $\max_{i \neq j} |\langle \varphi_i, \varphi_j \rangle|^2$

Welch bound

$$\max_{i \neq j} |\langle \varphi_i, \varphi_j \rangle|^2 \geq \frac{n-d}{d(n-1)}$$

With equality if and only if

- **Equiangular**: $|\langle \varphi_i, \varphi_j \rangle|^2 = b$ for all $i \neq j$
- **Tightness**: $\Phi\Phi^* = cI$

$\Big\}$ Φ is an ETF

Understanding Optimal Line Packings

Optimal line packings are understood in two ways

Geometrically as ETFs

- **Equiangular**: $i \neq j$
  $|\langle \varphi_i, \varphi_j \rangle|^2 = b$
- **Tightness**: $\Phi\Phi^* = cI$

Combinatorially with

$$b = \frac{n-d}{d(n-1)}$$

- $n = \#$ lines
- $d =$ dimension

Understanding Optimal Line Packings

Optimal line packings are understood in two ways

Geometrically as ETFs

- **Equiangular**: $i \neq j$
  $|\langle \varphi_i, \varphi_j \rangle|^2 = b$
- **Tightness**: $\Phi\Phi^* = cI$

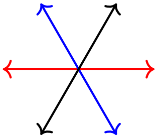
Combinatorially with

$$b = \frac{n-d}{d(n-1)}$$

- $n = \#$ lines
- $d =$ dimension

Example: Optimal line packing in $\mathbb{R}^2$ (an ETF)

$$\Phi = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$$



$$\theta = \frac{2\pi}{3} \quad \text{and} \quad b = |-1/2|^2 = 1/4 = \frac{3-2}{2(3-1)}$$

Understanding Optimal Line Packings

Optimal line packings are understood in two ways

Geometrically as ETFs

- **Equiangular**: $i \neq j$ $|\langle \varphi_i, \varphi_j \rangle|^2 = b$
- **Tightness**: $\Phi\Phi^* = cI$

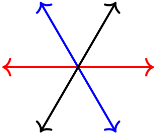
Combinatorially with

$$b = \frac{n-d}{d(n-1)}$$

- $n = \#$ lines
- $d =$ dimension

Example: Optimal line packing in $\mathbb{R}^2$ (an ETF)

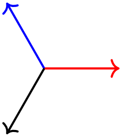
$$\Phi = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$$



$$\theta = \frac{2\pi}{3} \quad \text{and} \quad b = |-1/2|^2 = 1/4 = \frac{3-2}{2(3-1)}$$

Goal of talk: Do this but over finite fields.

Line Packings over Finite Fields



Discretizing Reality: Finite Field Analog to $\mathbb{R}^d$

Real IP Spaces	$\rightsquigarrow$	**Orthogonal Geometries**
$\mathbb{R}^d$	$\rightsquigarrow$	$\mathbb{F}_q^d$, where $q = p^\ell$ is odd.

Inner Products	$\rightsquigarrow$	**Non-Degenerate Scalar Products**

Discretizing Reality: Finite Field Analog to $\mathbb{R}^d$

Real IP Spaces	$\rightsquigarrow$	**Orthogonal Geometries**
$\mathbb{R}^d$	$\rightsquigarrow$	$\mathbb{F}_q^d$, where $q = p^\ell$ is odd.

Inner Products	$\rightsquigarrow$	**Non-Degenerate Scalar Products**
$\langle -, - \rangle : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$		
$\langle x, y \rangle = \langle y, x \rangle$		
$\langle x, - \rangle : \mathbb{R}^d \to \mathbb{R}$ linear		

Discretizing Reality: Finite Field Analog to $\mathbb{R}^d$

Real IP Spaces	$\rightsquigarrow$	**Orthogonal Geometries**
$\mathbb{R}^d$	$\rightsquigarrow$	$\mathbb{F}_q^d$, where $q = p^\ell$ is odd.

Inner Products	$\rightsquigarrow$	**Non-Degenerate Scalar Products**
$\langle -, - \rangle : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$		
$\langle x, y \rangle = \langle y, x \rangle$		
$\langle x, - \rangle : \mathbb{R}^d \to \mathbb{R}$ linear		
$\langle x, x \rangle > 0$ iff $x \neq 0$		

Discretizing Reality: Finite Field Analog to $\mathbb{R}^d$

Real IP Spaces	$\rightsquigarrow$	**Orthogonal Geometries**
$\mathbb{R}^d$	$\rightsquigarrow$	$\mathbb{F}_q^d$, where $q = p^\ell$ is odd.

Inner Products	$\rightsquigarrow$	**Non-Degenerate Scalar Products**
$\langle -, - \rangle : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$		$\langle -, - \rangle : \mathbb{F}_q^d \times \mathbb{F}_q^d \to \mathbb{F}_q$
$\langle x, y \rangle = \langle y, x \rangle$	$\rightsquigarrow$	$\langle x, y \rangle = \langle y, x \rangle$
$\langle x, - \rangle : \mathbb{R}^d \to \mathbb{R}$ linear		$\langle x, - \rangle : \mathbb{F}_q^d \to \mathbb{F}_q$ linear
$\langle x, x \rangle > 0$ iff $x \neq 0$		

Discretizing Reality: Finite Field Analog to $\mathbb{R}^d$

Real IP Spaces	$\rightsquigarrow$	**Orthogonal Geometries**
$\mathbb{R}^d$	$\rightsquigarrow$	$\mathbb{F}_q^d$, where $q = p^\ell$ is odd.

Inner Products	$\rightsquigarrow$	**Non-Degenerate Scalar Products**
$\langle -, - \rangle : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$		$\langle -, - \rangle : \mathbb{F}_q^d \times \mathbb{F}_q^d \to \mathbb{F}_q$
$\langle x, y \rangle = \langle y, x \rangle$	$\rightsquigarrow$	$\langle x, y \rangle = \langle y, x \rangle$
$\langle x, - \rangle : \mathbb{R}^d \to \mathbb{R}$ linear		$\langle x, - \rangle : \mathbb{F}_q^d \to \mathbb{F}_q$ linear
$\langle x, x \rangle > 0$ iff $x \neq 0$	$\rightsquigarrow$	$\langle x, y \rangle \neq 0$ for some $y \in \mathbb{F}_q^d$ iff $x \neq 0$

Discretizing Reality: Finite Field Analog to $\mathbb{R}^d$

Real IP Spaces	$\rightsquigarrow$	Orthogonal Geometries
$\mathbb{R}^d$	$\rightsquigarrow$	$\mathbb{F}_q^d$, where $q = p^\ell$ is odd.

Inner Products	$\rightsquigarrow$	Non-Degenerate Scalar Products
$\langle -, - \rangle : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$		$\langle -, - \rangle : \mathbb{F}_q^d \times \mathbb{F}_q^d \to \mathbb{F}_q$
$\langle x, y \rangle = \langle y, x \rangle$	$\rightsquigarrow$	$\langle x, y \rangle = \langle y, x \rangle$
$\langle x, - \rangle : \mathbb{R}^d \to \mathbb{R}$ linear		$\langle x, - \rangle : \mathbb{F}_q^d \to \mathbb{F}_q$ linear
$\langle x, x \rangle > 0$ iff $x \neq 0$	$\rightsquigarrow$	$\langle x, y \rangle \neq 0$ for some $y \in \mathbb{F}_q^d$ iff $x \neq 0$

Example: Non-degeneracy as a proof of being non-zero

$V = \mathbb{F}_3^3$ with $\langle x, y \rangle = x^\intercal y$ the dot product.

$$x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\rangle = 0 \quad \left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle = 1$$

Discretizing Reality: Two Types of Orthogonal Geometries

Definition

A $\mathbb{F}_q$-vector space V is called non-degenerate if it has a non-degenerate scalar product. V is an orthogonal geometry.

$V = \mathbb{F}_q^d$, with $\langle x, y \rangle = x^\intercal M y$, where $M = M^\intercal$ and is invertible

Discretizing Reality: Two Types of Orthogonal Geometries

Definition

A $\mathbb{F}_q$-vector space V is called non-degenerate if it has a non-degenerate scalar product. V is an orthogonal geometry.

$V = \mathbb{F}_q^d$, with $\langle x, y \rangle = x^\intercal M y$, where $M = M^\intercal$ and is invertible

Classification: An orthogonal geometry V with $\langle x, y \rangle = x^\intercal M y$

- $\det M$ a square (i.e. $\exists z \in \mathbb{F}_q$, $\det M = z^2$)
- $\det M$ not a square

Example: Non-square determinant

$V = \mathbb{F}_3^4$ with $\langle x, y \rangle = x^\intercal M y$, where $M = \mathrm{Diag}(1, 1, 1, 2)$

$$\left\langle \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} \right\rangle = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} = 1$$

Plato's Allegory of an Inner Product

Inner Product Spaces:

Plato's Allegory of an Inner Product

Inner Product Spaces:

- $\Phi = \left[\varphi_1, \ldots, \varphi_n\right]$ and its Gram matrix $\Phi^*\Phi = [\langle \varphi_i, \varphi_j \rangle]$ give equivalent information.
- Subspaces of inner product spaces are inner product spaces

Plato's Allegory of an Inner Product

Inner Product Spaces:

- $\Phi = \left[\varphi_1, \ldots, \varphi_n\right]$ and its Gram matrix $\Phi^*\Phi = [\langle \varphi_i, \varphi_j \rangle]$ give equivalent information.
- Subspaces of inner product spaces are inner product spaces

Orthogonal Geometries: Not the case. Consider an orthogonal geometry $V = \mathbb{F}_3^4$ with $\langle x, y \rangle = x^\intercal y$

$$\Phi = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 2 & 2 \\ 0 & 1 \end{bmatrix} \quad \Phi^\dagger \Phi = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \mathrm{im}\,\Phi \subseteq V \text{ is degenerate.}$$

Frame Theory (Greaves, Iverson, Jasper, Mixon; 2022), (J 2025)

Let $\Phi = \left[\varphi_1, \varphi_2 \ldots, \varphi_n\right] \in \mathbb{F}_q^{d \times n}$, $a, b, c \in \mathbb{F}_q$. Then Φ is a

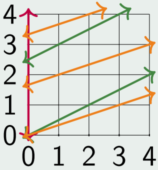
- **frame** for $\operatorname{im}\Phi$ if $\operatorname{im}\Phi$ is non-degenerate $\Leftrightarrow \mathrm{rk}(\Phi) = \mathrm{rk}(\Phi^\dagger \Phi)$
- c-**tight frame** for $\operatorname{im}\Phi$ if $\Phi\Phi^\dagger\Phi = c\Phi$
- (a, b)-**equiangular** if
  - $\langle \varphi_j, \varphi_j \rangle = a$ for all j
  - $\langle \varphi_j, \varphi_k \rangle^2 = b$ for all $j \neq k$
- (a, b, c)-**equiangular tight frame(ETF)** if all the above.

Let $\Phi = \left[\varphi_1, \varphi_2 \ldots, \varphi_n\right] \in \mathbb{F}_q^{d \times n}$, $a, b, c \in \mathbb{F}_q$. Then Φ is a

- **frame** for $\operatorname{im}\Phi$ if $\operatorname{im}\Phi$ is non-degenerate $\Leftrightarrow \mathrm{rk}(\Phi) = \mathrm{rk}(\Phi^\dagger \Phi)$
- c-**tight frame** for $\operatorname{im}\Phi$ if $\Phi\Phi^\dagger\Phi = c\Phi$
- (a, b)-**equiangular** if
  - $\langle \varphi_j, \varphi_j \rangle = a$ for all j
  - $\langle \varphi_j, \varphi_k \rangle^2 = b$ for all $j \neq k$
- (a, b, c)-**equiangular tight frame(ETF)** if all the above.

Example: $V = \mathbb{F}_5^2$ with $\langle x, y \rangle = x^\intercal M y$, where $M = \mathrm{Diag}(1, 3)$

$$\Phi = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 1 & 1 \end{bmatrix} \qquad \Phi^\dagger\Phi = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$



Φ is an $(2, 1, 3)$-ETF for $\mathbb{F}_5^2$ of $n = 3$ vectors.

Frame Theory: 4×10 ETF

Example (Greaves, Iverson, Jasper, Mixon 2022)

$V = \mathbb{F}_3^4$ with $\langle x, y \rangle = x^\intercal M y$, where $M = \mathrm{Diag}(1, 1, 1, 2)$

$$\Phi = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 2 & 2 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 2 & 1 & 2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Φ is an $(0, 1, 0)$-ETF for $\mathbb{F}_3^4$ of $n = 10$ vectors.

Frame Theory: 4×10 ETF

Example (Greaves, Iverson, Jasper, Mixon 2022)

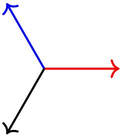
$V = \mathbb{F}_3^4$ with $\langle x, y \rangle = x^\intercal M y$, where $M = \mathrm{Diag}(1, 1, 1, 2)$

$$\Phi = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 2 & 2 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 2 & 1 & 2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Φ is an $(0, 1, 0)$-ETF for $\mathbb{F}_3^4$ of $n = 10$ vectors.

- Φ is a maximal ETF for $\mathbb{F}_3^4$
- No 4×10 real ETF is known to exist
- Contains 30 regular 3-simplices: 15 square geometry, 15 non-square geometry, both pairs of 15 form $(10, 4, 2)$-BIBDs

The Welch Bound Revisited



On the Failure of a Welch Bound

Theorem (Greaves, Iverson, Jasper, Mixon; 2022)

If $\Phi \in \mathbb{F}_q^{d \times n}$ is a (a, b, c)-ETF then $d(n-1)b = (n-d)a^2$
(if the field is nice: $b = \frac{n-d}{d(n-1)} a^2$)

On the Failure of a Welch Bound

Theorem (Greaves, Iverson, Jasper, Mixon; 2022)

If $\Phi \in \mathbb{F}_q^{d \times n}$ is a (a, b, c)-ETF then $d(n-1)b = (n-d)a^2$
(if the field is nice: $b = \frac{n-d}{d(n-1)} a^2$)

Example: $V = \mathbb{F}_5^7$ with $\langle x, y \rangle = x^\intercal y$

$$\Phi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 & 4 & 0 & 2 \\ 0 & 1 & 1 & 2 & 1 & 2 & 2 & 3 \\ 1 & 0 & 1 & 2 & 3 & 2 & 2 & 3 \\ 1 & 1 & 0 & 2 & 3 & 4 & 4 & 1 \end{bmatrix}$$

Φ is an $(2, 1)$-equiangular frame for V.
It satisfies $b \equiv 1 \equiv \frac{1}{49} 2^2 \equiv \frac{n-d}{d(n-1)} a^2$.

On the Failure of a Welch Bound

Theorem (Greaves, Iverson, Jasper, Mixon; 2022)

If $\Phi \in \mathbb{F}_q^{d\times n}$ is a (a,b,c)-ETF then $d(n-1)b = (n-d)a^2$
(if the field is nice: $b = \frac{n-d}{d(n-1)}a^2$)

Example: $V = \mathbb{F}_5^7$ with $\langle x, y\rangle = x^\intercal y$

$$\Phi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 & 4 & 0 & 2 \\ 0 & 1 & 1 & 2 & 1 & 2 & 2 & 3 \\ 1 & 0 & 1 & 2 & 3 & 2 & 2 & 3 \\ 1 & 1 & 0 & 2 & 3 & 4 & 4 & 1 \end{bmatrix}$$

Φ is an $(2,1)$-equiangular frame for V.
It satisfies $b \equiv 1 \equiv \frac{1}{49}2^2 \equiv \frac{n-d}{d(n-1)}a^2$. But Φ is not a tight frame

On the Failure of a Welch Bound

Theorem (Greaves, Iverson, Jasper, Mixon; 2022)

If $\Phi \in \mathbb{F}_q^{d\times n}$ is a (a,b,c)-ETF then $d(n-1)b = (n-d)a^2$
(if the field is nice: $b = \frac{n-d}{d(n-1)}a^2$)

Example: $V = \mathbb{F}_5^7$ with $\langle x, y\rangle = x^\intercal y$

$$\Phi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 & 4 & 0 & 2 \\ 0 & 1 & 1 & 2 & 1 & 2 & 2 & 3 \\ 1 & 0 & 1 & 2 & 3 & 2 & 2 & 3 \\ 1 & 1 & 0 & 2 & 3 & 4 & 4 & 1 \end{bmatrix} \quad \Phi\Phi^\dagger\Phi = \begin{bmatrix} 3 & 3 & 2 & 3 & 3 & 3 & 2 & 4 \\ 3 & 3 & 2 & 3 & 4 & 0 & 3 & 1 \\ 3 & 4 & 2 & 4 & 1 & 3 & 3 & 1 \\ 4 & 0 & 1 & 0 & 1 & 3 & 1 & 3 \\ 2 & 1 & 4 & 2 & 0 & 2 & 1 & 3 \\ 0 & 3 & 2 & 0 & 1 & 4 & 4 & 1 \\ 1 & 0 & 0 & 1 & 3 & 0 & 1 & 0 \end{bmatrix}$$

Φ is an $(2,1)$-equiangular frame for V.
It satisfies $b \equiv 1 \equiv \frac{1}{49}2^2 \equiv \frac{n-d}{d(n-1)}a^2$. But Φ is not a tight frame

A New Hope: Using Sums of Triple Products

- Triple Product: $\Delta(\varphi_j, \varphi_k, \varphi_\ell) = \langle \varphi_j, \varphi_k \rangle \langle \varphi_k, \varphi_\ell \rangle \langle \varphi_\ell, \varphi_j \rangle$
- Sums of triple products have been used to study the algebraic properties of frames by (Appleby et. al.; 2011), (Zhu; 2015), and (King; 2019).

A New Hope: Using Sums of Triple Products

- Triple Product: $\Delta(\varphi_j, \varphi_k, \varphi_\ell) = \langle \varphi_j, \varphi_k \rangle \langle \varphi_k, \varphi_\ell \rangle \langle \varphi_\ell, \varphi_j \rangle$
- Sums of triple products have been used to study the algebraic properties of frames by (Appleby et. al.; 2011), (Zhu; 2015), and (King; 2019).

Let $\mathbb{F}_q$ be a field with $q = p^\ell$ elements, $p \nmid dn$

Theorem (J)

Let $\Phi = \left[\varphi_1, \ldots, \varphi_n\right] \in \mathbb{F}_q^{d \times n}$ be an (a, b)-equiangular frame for $\mathbb{F}_q^d$ $(a \neq 0)$. Then Φ is an $(a, b, na/d)$ETF if and only if

-
-

A New Hope: Using Sums of Triple Products

- Triple Product: $\Delta(\varphi_j, \varphi_k, \varphi_\ell) = \langle \varphi_j, \varphi_k \rangle \langle \varphi_k, \varphi_\ell \rangle \langle \varphi_\ell, \varphi_j \rangle$
- Sums of triple products have been used to study the algebraic properties of frames by (Appleby et. al.; 2011), (Zhu; 2015), and (King; 2019).

Let $\mathbb{F}_q$ be a field with $q = p^\ell$ elements, $p \nmid dn$

Theorem (J)

Let $\Phi = \left[\varphi_1, \dots, \varphi_n\right] \in \mathbb{F}_q^{d \times n}$ be an (a, b)-equiangular frame for $\mathbb{F}_q^d$ $(a \neq 0)$. Then Φ is an $(a, b, na/d)$ETF if and only if

- $d(n-1)b = (n-d)a^2$
-

A New Hope: Using Sums of Triple Products

- Triple Product: $\Delta(\varphi_j, \varphi_k, \varphi_\ell) = \langle \varphi_j, \varphi_k \rangle \langle \varphi_k, \varphi_\ell \rangle \langle \varphi_\ell, \varphi_j \rangle$
- Sums of triple products have been used to study the algebraic properties of frames by (Appleby et. al.; 2011), (Zhu; 2015), and (King; 2019).

Let $\mathbb{F}_q$ be a field with $q = p^\ell$ elements, $p \nmid dn$

Theorem (J)

Let $\Phi = \left[\varphi_1, \ldots, \varphi_n\right] \in \mathbb{F}_q^{d\times n}$ be an (a,b)-equiangular frame for $\mathbb{F}_q^d$ $(a \neq 0)$. Then Φ is an $(a, b, na/d)$ETF if and only if

- $d(n-1)b = (n-d)a^2$
- $\displaystyle\sum_{\ell=1}^{n} \langle \varphi_j, \varphi_k \rangle \langle \varphi_k, \varphi_\ell \rangle \langle \varphi_\ell, \varphi_j \rangle = \frac{nab}{d}$ for all $j \neq k$

A New Hope: Using Sums of Triple Products

- Triple Product: $\Delta(\varphi_j, \varphi_k, \varphi_\ell) = \langle \varphi_j, \varphi_k \rangle \langle \varphi_k, \varphi_\ell \rangle \langle \varphi_\ell, \varphi_j \rangle$
- Sums of triple products have been used to study the algebraic properties of frames by (Appleby et. al.; 2011), (Zhu; 2015), and (King; 2019).

Let $\mathbb{F}_q$ be a field with $q = p^\ell$ elements, $p \nmid dn$

Theorem (J)

Let $\Phi = \left[\varphi_1, \ldots, \varphi_n\right] \in \mathbb{F}_q^{d \times n}$ be an (a, b)-equiangular frame for $\mathbb{F}_q^d$ $(a \neq 0)$. Then Φ is an $(a, b, na/d)$ETF if and only if

- $d(n-1)b = (n-d)a^2$
- $\displaystyle\sum_{\ell=1}^{n} \langle \varphi_j, \varphi_k \rangle \langle \varphi_k, \varphi_\ell \rangle \langle \varphi_\ell, \varphi_j \rangle = \frac{nab}{d}$ for all $j \neq k$

- This theorem is also true for unitary geometries.
- For certain finite fields, Φ need not be a frame.
- Works for any field, not just finite fields.

Additional Results

On the Structure of Frames and Equiangular Lines over Finite Fields and their Connections to Design Theory (arXiv:2505.12175)

- For ETFs, containing regular simplices is more or less determined by collections of vectors having equal triple products.

Additional Results

On the Structure of Frames and Equiangular Lines over Finite Fields and their Connections to Design Theory (arXiv:2505.12175)

- For ETFs, containing regular simplices is more or less determined by collections of vectors having equal triple products.
- Expanded on the theory of Naimark complements from (Greaves, Iverson, Jasper, Mixon; 2022), showing that in general $\Phi^\dagger \Phi + \Psi^\dagger \Psi = cI$ is not sufficient and an additional condition is needed.

Additional Results

On the Structure of Frames and Equiangular Lines over Finite Fields and their Connections to Design Theory (arXiv:2505.12175)

- For ETFs, containing regular simplices is more or less determined by collections of vectors having equal triple products.
- Expanded on the theory of Naimark complements from (Greaves, Iverson, Jasper, Mixon; 2022), showing that in general $\Phi^\dagger\Phi + \Psi^\dagger\Psi = cI$ is not sufficient and an additional condition is needed.
- Generalized Gillespie incoherent sets, showing ETFs in orthogonal geometries that saturated a incoherence bound, often give rise to quasi-symmetric 2-designs, and 4-designs in special cases.

Questions

$$\Phi = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

An $(0, 1, 1)$-ETF for $\mathbb{F}_3^3$