A motivating Example: Symmetric (Self adjoint) matricies

Ex dot product $\langle u, v\rangle = u^t v$ where $\langle Av, u\rangle = (Au)^t = u^t A^t u = \langle u, A^t u\rangle$

- Self adjoint: $A^t = A$

These matricies aren't closed under mult.

$$(AB)^t = B^t A^t = BA \overset{?}{=} AB$$

Solution: Invent a new product

$$A.B = \tfrac{1}{2}(AB + BA) \quad \text{where } (A.B)^t = \tfrac{1}{2}\left((AB)^t + (BA)^t\right) = A^t . B^t = A.B$$

↳ this work for any asoc. algebra: special Jordan algebra

A Jordan algebra J is an algebra over K with product $a.b$ Satisfying $a.b = b.a$ and $(a^{.2}.b).a = a^{.2}.(b.a)$

If my architypical example is that of matries They should act on Vectors! ie I should have a module!

let ${}_J M_J$ be a Jordan bimodule for M a K-Vec Space

Ex golden Rule of repn theory: act on your self

So ${}_J J_J$ is a J bimodule with $a.m = m.a$

→ really This can be thought of as a righ module but

$$J \times J \to J \quad \overset{?}{\longleftrightarrow} \quad J \xrightarrow{\rho} Hom_K(J, J)$$

$a.(b.m) \neq (a.b).m$ Not a hom of Jordan algebras! This map into $Hom_K(J,J)$ an asoc. algebra

This ρ is an example of a multiplication Specialization.

Same with $J \to Hom_K(M, M)$

A multiplication Specialization of J in A (an assoc alg) is lin map $\rho: J \to A$ s.t. $[a^\rho, a^{.2\rho}] = 0$ and $2a^\rho b^\rho a^\rho + (a^{.2}.b)^\rho = 2a^\rho (a.b)^\rho + b^\rho (a^{.2})^\rho$

The "best" mult. Spec.: An asoc alg $U(J)$ w/ mult. Spec ρ_u is a universal multiplication envelope of J if

$(\forall\exists)$

$$J \xrightarrow{\rho_u} U(J) \quad \Bigg| \quad \forall \quad \begin{array}{ccc} J & \xrightarrow{\rho_u} & U(J) \\ \downarrow \rho & & \\ A & & \end{array} \quad \Bigg| \quad \exists! \quad \begin{array}{ccc} J & \xrightarrow{\rho_u} & U(J) \\ \downarrow \rho & \swarrow \mu & \\ A & & \end{array}$$

Pierce decompositions

Recal : A an asoc. alg w/1 with e an idempotent
then $\{e, 1-e\}$ was supp. orthog collection of idempotents
$A = 1 \cdot A \cdot 1 = (e+(1-e))A(e+(1-e)) = eAe \oplus eA(1-e) \oplus (1-e)Ae \oplus (1-e)A(1-e)$
was the pierce decomposition of A.

The help us do the same thing for Jordan algebras
lets introduce a notion of a triple product.
we have some choice here but the "correct" notion is
$\{abc\} = (a.b).c + a.(b.c) - (a.c).b$
$\rightarrow$ if $a.b = \frac{1}{2}(ab+ba)$ then $\{abc\} = \frac{1}{2}(abc+cba)$
This gives us two maps $U_{a,b} : x \mapsto \{axb\}$ and
$U_a = U_{a,a} : x \mapsto \{axa\}$ both live in $Hom_k(J,J)$
and $U_{a,b}x = \{axb\} = \{bxa\} = U_{b,a}x$

let e be an idempotent of J w/1 ie $e.e = e$ then
$\{e, 1-e\}$ is a collection of supp orthog idempotents in J
Then with a similar trick
$J = \{1J1\} \stackrel{?}{=} \{eJe\} \oplus \{eJ(1-e)\} \oplus \{(1-e)Je\} \oplus \{(1-e)J(1-e)\} = \{eJe\} \oplus 2\{eJ(1-e)\} \oplus \{(1-e)J(1-e)\}$

In general $e_1, \dots, e_n$ collection orthog supp idempotents
Then let $J_{ii} = JU_{e_i} = \{e_i J e_i\}$ and
$J_{ij} = JU_{e_i,e_j}2 = 2\{e_i J e_j\}$ for $i \neq j$
Then
$$J = \bigoplus_{i \leq j} J_{ij}$$ is the pierce decomp of J
relative to the e_i

let $P_{ii} = U_{e_i}$ and $P_{ij} = U_{e_i,e_j}2 = 2U_{e_i,e_j}$
Then $J_{ij} = JP_{ij}$ but importantly these P_{ij}'s
$\{P_{ij}\}_{i \leq j}$ is a collection of supp. orthog. idempotents
of $Hom_k(J,J)$