Something Something Jordan Pairs, and Matrix Multiplication

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1 Introduction

This paper is meant as a way of documenting my understanding of algebra with an emphasis on how and why algebra is the way it is, and so this document may be obvious or even outright insulting to a trained algebraist. I've come to the conclusion that algebra is a 4+ step process with a heavy amount of over lap between the steps:

Step 1) Understanding Examples: Find examples of algebra in the real word.

Step 2) Building Algebra: Using your examples to (a) find operators, or a grammar (b) find identities that your examples follow and make them the laws of your algebra. (c) do Noether's Isomorphism theorem connecting congruences, homomorphisms and quotients.

Step 3) Studying Algebra: Study your newly built algebra

Step 4) Repeat: Study your examples again with new insights or build new algebra from the algebra you created in the previous steps

This paper is roughly outlined to follow these steps for Jordan Pairs and Jordan Algebras. The following definitions and theorems come from the book Jordan Pairs by Ottmar Loos.

1.1 Step 1: A Motivating Example: Matrices with Geometry

Matrices are cool. Thats it! Thats all the motivation I need. Groups can be used to talk about the algebra of multiplying invertible matrices with common dimensions, with the General Linear group. Groupoids can be used to work with all square matrices. And a Category can be used to study the behavior of matrix multiplication in general. However these three algebra leave gaps, categories and groupoids can often be too broad, allowing for partially defined multiplications while groups and groupoids are too narrow, allowing only invertible matrices.

A motivating example for Jordan Pairs

Often one may want to study rectangular matrices. For example Let V^+ be the k-module of $q \times p$ matrices $M_{q,p}(k)$ and V^- be the k-module of $p \times q$ matrices $M_{p,q}(k)$. These matrices are clearly compatible under multiplication but there is a problem! If $A \in V^+$ and $B \in V^-$, then their product is in neither of the original modules. This requires a new operator, a trinary operation where if $A, C \in V^+$ and $B \in V^-$ then $ABC \in V^+$ and vice-versa. In fact this trinary operator would be tri-linear. This would give us a nice algebra, but it has a problem, it does not behave nicely with geometry. Geometry often looks at bilinear forms which can be defined by a transpose. For an multiplication operator to "behave nicely" often means that that the transpose respects multiplication (that is the transpose and the triple multiplication should be commuting operators). However notice with this simple triple product, we have that ABC is bilinear in A and C but notice that under a transpose $(ABC)^t = C^tB^tA^t$ we have a commuting issue. And so instead we want to define an operator that is symmetric in A and C. An easy way to do this is to define an operator $\{A, B, C\} = ABC + CBA$. This gives us $\{A, B, C\}C^tB^tA^t + = A^tB^tC^t = \{A^t, B^t, C^t\}$ as desired.

A motivating example for Jordan Algebras

A simpler example of this commuting issue is with symmetric(self-adjoint) matrices which is our second motivating example, where $A = A^t$ and $B = B^t$. If we had a bilinear form: $\langle x|y\rangle = x^ty$ we could study the self adjoint elements $\langle Ax|y\rangle = \langle x|A^ty\rangle$. But we run into the exact same problem: that symmetric matrices are generally not closed under multiplication $(AB)^t = B^t A^t = BA$. And restricting the study of matrices to commutative ones is not a reasonable ask! So again the fix is to invent a new matrix multiplication that is symmetric and still does all the same things.

1.2 Step 2: Fumbling to a Definition

1.2.1 Jordan Pairs

We are not saying Ottmar Loos Fumbled to the definitions we will present below (They are simply too nice and also intricate for that to be the case) but at some point to do algebra you have to just fumble to something that looks good enough, and hopefully in retrospect you can convince yourself why your algebra is a good one.

As mentioned above we want to create an algebra that looks at how pairs of modules interact, through a symmetric trilinear operator that we will think about as being our triple matrix product symmetricified. So we need to define our two k-modules with our trilinear operator that goes between them

$$\begin{aligned}
\langle\langle V^+\rangle\rangle ::= {} & 0_+ \\
& |\,\langle V^+\rangle + \langle V^+\rangle \\
& |\,-\langle V^+\rangle \\
& |\,\langle k\rangle \cdot \langle V^+\rangle \\
& |\{\langle V^+\rangle, \langle V^-\rangle, \langle V^+\rangle\}
\end{aligned}$$

$$\begin{aligned}
\langle\langle V^-\rangle\rangle ::= {} & 0_- \\
& |\,\langle V^-\rangle + \langle V^-\rangle \\
& |\,-\langle V^-\rangle \\
& |\,\langle k\rangle \cdot \langle V^-\rangle \\
& |\{\langle V^-\rangle, \langle V^+\rangle, \langle V^-\rangle\}
\end{aligned}$$

And then we can define the grammar for the algebra of the pair of modules. This is the signature for a Jordan pair over k

$$\langle\langle \mathrm{JP}_k\rangle\rangle ::= \langle V^+\rangle \,|\, \langle V^-\rangle$$

Now that we know what operators we have in our world we need laws to tell us what these operators mean and how we can rewrite expressions. First we need the laws that tell us that V^+ and V^- are in fact k-modules. And we also need the laws that tell about our triple product, that $\{-,-,-\} : V^\sigma \times V^{-\sigma} \times V^\sigma \to V^\sigma$ is a trilinear operator and symmetric in the first and third terms.

From these laws we can create additional operators. From the equivalence of symmetric bilinear maps and quadratic maps we can define two quadratic map for $\sigma = \pm$ where

$$Q_\sigma(-) : V^\sigma \to \hom(V^{-\sigma}, V^\sigma) \text{ which are given by } 2Q_\sigma(x)y = \{xyx\}$$

We can also create their corresponding symmetric bilinear maps (which recover the original $\{-,-,-\}$ product.) $Q_\sigma(-,-) : V^\sigma \times V^\sigma \to \hom(V^{-\sigma}, V^\sigma)$ which are given by $Q_\sigma(x,z) = Q_\sigma(x+z) - Q_\sigma(x) - Q_\sigma(z)$.. Finally we will create the bilinear maps $D_\sigma(x,y)z = Q_\sigma(x,z)y = \{x,y,z\}$. Although these new operators are unnecessarily they help which much of the theory of Jordan Pairs in a more concise way. Notice that in the example in the section above we would have that $Q_\sigma(x)y = xyx$.

Now that we have our algebra with some initial laws we need to build some more laws using the formulas of JP_k. These laws are created to mirror the behavior of our motivating examples. We will build the following laws:

$$\{x, y, Q_\sigma(x)z\} = Q_\sigma(x)\{y, x, z\} \tag{JP1}$$

$$\{Q_\sigma(x)y, y, z\} = \{x, Q_{-\sigma}(y)x, z\} \tag{JP2}$$

$$Q_\sigma(Q_\sigma(x)y)z = Q_\sigma(x)(Q_{-\sigma}(y)(Q_\sigma(x)z)) \tag{JP3}$$

Combining these laws we have the following definition of a Jordan Pair

Definition 1.1. The pair $V = (V^+, V^-)$ is called a Jordan pair if $V \vDash_{\mathrm{JP}_k} \mathcal{L}$ where $\mathcal{L}$ are the laws outlined above. More Concisely we may say that a pair $V = (V^+, V^-)$ of k-modules along with a pair of quadratic maps (Q_+, Q_-), which define the trilinear operator $\{-,-,-\}$, is called a Jordan pair if JP1-JP3 hold in all scalar extensions of V

$$\{x, y, Q_\sigma(x)z\} = Q_\sigma(x)\{y, x, z\} \tag{JP1}$$

$$\{Q_\sigma(x)y, y, z\} = \{x, Q_{-\sigma}(y)x, z\} \tag{JP2}$$

$$Q_\sigma(Q_\sigma(x)y)z = Q_\sigma(x)(Q_{-\sigma}(y)(Q_\sigma(x)z)) \tag{JP3}$$

Note that we will often represent an element of a Jordan Pair as $x \in V$ without directly specifying which module x is a member of. And in general a pair of objects signifies that objects may come from either of the k-modules. Now that we have a definition for our algebra we can define what it means to have sub-objects, homomorphisms, congruences, quotients and ideals. None of the following are particularly surprising (You see what I did there? Im acting like a trained algebraist)

Definition 1.2. A homomorphism $h : V \to W$ of Jordan Pairs is a pair of maps (h_+, h_-) such that $h_\sigma : V^\sigma \to W^\sigma$ is a k-module homomorphism such that the following holds

$$h_\sigma(\{x, y, z\}) = \{h_\sigma(x), h_{-\sigma}(y), h_\sigma(z)\} \tag{1}$$

This condition is equivalent to

$$h_\sigma(Q_\sigma(x)y) = Q_\sigma(h_\sigma(x))h_{-\sigma}(y) \tag{2}$$

Definition 1.3. Let $h = (h_+, h_-)$ be a homomorphism of Jordan Pairs then the kernel of h is the pair $\ker(h) = (\{x \equiv z \pmod h | h_+(x) = h_+(z)\}, \{x \equiv z \pmod h | h_-(x) = h_-(z)\})$ More concisely we will write $\ker(h) = \{x \equiv z \pmod h | h(x) = h(z)\} \subseteq V \times V$

The kernel of a homomorphism $h : V \to W$ is the resulting congruence on V that relates elements by their outputs in the homomorphism h. We know from Noether's Isomorphism theorem that the kernels of V represent all the congruences on V.

Because of the underlying module structure of V^σ we have the following observation for congruence of V: that $x \equiv z \Leftrightarrow x - z \equiv 0_\sigma$. This means that congruences of Jordan Pairs can be identified with the sub-object $()\{x \in V | x \equiv 0_\sigma\}, \{x \in V | x \equiv 0_\sigma\}) \subseteq V$ up to univalence. In the context of kernels this means that $(x, z) \in \ker(h)$ if and only if $h_\sigma(x - z) = 0_\sigma$. This means the Kernel of a homomorphism of Jordan Pairs can be identified with $\{x \in V | h_\sigma(x) = 0_\sigma\}$. We will call this type of sub-object an Ideal of V.

Definition 1.4. $I = (I^+, I^-)$ is an ideal of V if there exists a congruence $\equiv$ on V such that $x \equiv z$ if and only if $x - z \in I$. That is I is an ideal if it can be identified with some congruence on V. In terms of kernels this means that $I = (I^+, I^-)$ is an ideal of V if there exists a homomorphism $h : V \to W$ such that $(x, z) \in \ker(h)$ if and only if $x - z \in I$. That is I is an ideal if it can be identified with the kernel of some homomorphism.

Definition 1.5. Let V be a Jordan Pair and I an ideal then then quotient V/I is the partitions of V defined by the congruence corresponding to I. That is $V/I = (V^+/I^+, V^-/I^-) = (\{v + I^+ | v \in V^+\}, \{v + I^- | v \in V^-\})$

We will henceforth associate congruences with their corresponding ideal, that is $\ker(h) = \{x \in V | h(x) = 0_\sigma\}$. This gives us a special case of Noethers Isomorphism theorem relating homomorphism, ideals and Quotients

Proposition 1.6. *Let $I \subseteq V$ then the following are equivalent*

- *I is an ideal of V*
- *V/I is a Jordan Pair*
- *$h : V \to V/I$ is a epimorphism with $\ker(h) = I$*

This allows us a characterization of Ideals of Jordan pairs using the the equivalence of Ideals and Quotients being Jordan Pairs, and so we need only check the conditions needed on I for V/I to be a Jordan Pair.

Proposition 1.7. *$I \subseteq V$ is an ideal of V if and only if $Q_\sigma(I^\sigma)V^{-\sigma} + Q_\sigma(V^\sigma)I^{-\sigma} + \{V^\sigma, V^{-\sigma}, I^\sigma\} \subseteq I^\sigma$*

An important part of step 3, is the study of simples. As an important observation is that any Jordan Pair can be the image of some homomorphism, or in other words any Jordan pair can be the quotient of another Jordan Pair. So the simple Jordan Pairs are those whose quotients do not result any any new Jordan pairs

Definition 1.8. A non-zero Jordan pair V is called simple if $(0_+, 0_-)$ and V are its only ideals.

To study our algebra it is often helpful to identify important objects in our examples, such as automorphisms, possibly including twisted automorphisms. So we define $\mathrm{aut}(V) = \{h : V \to V | h \text{ is a isomorphims}\} \leq \mathrm{GL}(V^+) \times \mathrm{GL}(V^-)$. One important example of an automorphism of matrix modules is the transpose. However the standard transpose is not an automorphism of Jordan pairs. This follows from the fact that $(-)^t : V^+ = M_{p,q}(k) \to V^- = M_{q,p}(k)$ and vice versa. This however introduces the notion of an anti-homomorphism or twisted homomorphism.

Definition 1.9. An antihomomorphism between Jordan pairs V and W is a homomorphism $\eta : (V^+, V^-) \to (W^-, W^+)$, that is a pair of k-module homomorphisms $\eta_\sigma : V^\sigma \to W^{-\sigma}$ such that $\eta_\sigma(Q_\sigma(x)y) = (Q_{-\sigma}(\eta_\sigma(x))\eta_{-\sigma}(y))$. We often denote W^{op} to be the Jordan Pair $W^{\mathrm{op}} = (W^-, W^+)$ with quadratic maps (Q_-, Q_+). Furthermore a twisted automorphism η is called an involution if $\eta_{-\sigma}\eta_\sigma$ is the identity map on V^σ.

There are many important examples of involutions in Jordan pairs one being a transpose.

Jordan Algebras from Jordan Pairs

A closely related algebra to the Jordan pair is the Jordan algebra. A jordan pair $V = (V^+, V^-)$ studies the interactions between two modules. Jordan algebras allow us to study each module somewhat independently (Although there is a rich study of Jordan Algebras unrelated to Jordan Pairs). That is we can define the multiplication of the elements x and y in V^σ as the triple product $x \circ y := \{xvy\}$ for some fixed v. This creates a symmetric but not necessarily associative multiplication

Definition 1.10. Let $V = (V^+, V^-)$ be a Jordan pair with $v \in V^-$ then V^+ is Jordan algebra, denoted V_v^+ with Jordan Product $x \circ y = \{xvy\}$ squaring operation $x^2 = Q_+(x)v = \frac{1}{2}\{xvx\}$ and quadratic U-operator $U_x = Q_+(x)Q_-(v)$

And likewise a definition for V_v^- can be given through V^{op}. In Jordan algebras with a unit 1 we have that $x^2 = U_x(1)$, but this is generally rare in Jordan algebra that result from Jordan Pairs. This construction provides a powerful connection between the study of Jordan pairs and quadratic Jordan algebras.

A homomorphism $h : V \to W$ of Jordan Pairs induces a homomorphism $h_\sigma : V_v^\sigma \to W^\sigma_{h_{-\sigma}(v)}$ on the

1.3 Step 3: Studying The Algebra

Jordan's Web

Now that we have created the algebra of Jordan Pairs we need to study it. An important notion to study is that of invertibility. At first glance this may be tricky as Jordan Pairs have no classical binary product. Instead Jordan Pairs have quadratic map $Q_\sigma(x)y$ that acts as a two-sided product: the product of x on both sides of y, at least this is the intuition that our example gives us. This gives us a notion of inverses, that of inverting the two-sided product $Q_\sigma(x)$. Furthermore an inverse x^{-1} of an element x should satisfy $Q_{-\sigma}(x^{-1})Q_\sigma(x)y = y$. And Using JP3 we can determine that $Q_\sigma(x)y = Q_\sigma(x)Q_{-\sigma}(x^{-1})Q_\sigma(x)y = Q_\sigma(Q_\sigma(x)x^{-1})y$. And so $Q_\sigma(x)x^{-1} = x$ and therefore we have the following definition

Definition 1.11. Let V be a Jordan Pairs then an element $u \in V^\sigma$ is called invertible if quadratic product $Q_\sigma(u) : V^{-\sigma} \to V^\sigma$ is invertible. In this case $u^{-1} = Q_\sigma(u)^{-1}(u)$

In which case we have that $Q_\sigma(u)^{-1} = Q_{-\sigma}(u^{-1})$ as desired and $(u^{-1})^{-1}$. Invertible elements are generally rare in Jordan Pairs but have a nice correspondence with unital Jordan Algebras

Proposition 1.12. *Let V be a Jordan Pair with invertible element $v \in V^-$ and $u = v^{-1} \in V^+$. Then the corresponding Jordan Algebras $J = V_v^+$ and $J' = V_u^-$ are unital with units u and v respectively. Furthermore J and J' are isomorphism with $Q_-(v) : J \to J'$ and inverse $Q_+(u) : J' \to J$. Additionally the map $(Id_J, Q_-(v)) : (J, J) \to (V^+, V^-)$ is an isomorphism of Jordan Pairs.*

This gives a powerful connection between the study of Jordan Pairs with invertible elements and Unital Jordan Algebras.

1.3.1 Centering in on the Numbers

A common technic in the study of algebra is that of extending coefficients. As an example consider a Ω-module V, where Ω has no required structure, given by the action of $\cdot : \Omega \times V \to V$ that respects the structure of V. However Ω

may not represent all the actions of V and so often it is helpful to extent Ω to include all the actions of V that respect the structure of V and the action of Ω. Notice also that actions that preserve the original actions of Ω commute, and so such an extension of scalar comes down to a collection of commuting requirements. These new scalars are the endomorphisms $\mathrm{End}(_\Omega V)$ endowing V with a $\mathrm{End}(_\Omega V)$-module with nicer properties.

Let $V = (V^+, V^-)$ be a Jordan pair over k, where each V^σ is a k-module. Consider first $\mathrm{End}(V^+) \times \mathrm{End}(V^-)$, the endomorphisms of each V^σ preserving the k-module structures, which contains all automorphism of V. However for a map $a = (a_+, a_-) \in \mathrm{End}(V^+) \times \mathrm{End}(V^-)$ to respect the structure of V we would need the following commuting requirements First we need that the action of a commutes with the two-sided quadratic map in the sense that $a_\sigma Q_\sigma(x)y = Q_\sigma(x)a_{-\sigma}(y)$. Similarly we need that $a_\sigma(\{x, y, z\}) = \{x, y, a_\sigma(z)\}$. These requirements define what is know as the centroid.

Definition 1.13. Let V be a Jordan Pair over k. The Centroid of V denoted as $Z(V)$ is the collection of elements $a = (a_+, a_-) \in \mathrm{End}(V^+) \times \mathrm{End}(V^-)$ such that

$$a_\sigma Q_\sigma(x)y = Q_\sigma(x)a_{-\sigma}(y) \tag{Z1}$$

$$a_\sigma(\{x, y, z\}) = \{x, y, a_\sigma(z)\} \tag{Z2}$$

$$Q_\sigma(a_\sigma(x)) = a_\sigma^2 Q_\sigma(x) \tag{Z3}$$

Notice that requirement Z3 is only needed in the case of characteristic 2, otherwise Z3 follows from Z2 and the symmetry of the trilinear operator. And likewise the identities $a_\sigma(\{x, y, z\}) = \{a_\sigma(x), y, z\}$ and $a_\sigma(\{x, y, z\}) = \{x, a_{-\sigma}(y), z\}$ follow from the above.

The centroid represents the possible actions on V that preserve its structure, meaning the centroid may be an extension of the underlying scalars from k. To see this Notice that any scalar $\lambda \in k$ has the corresponding element $(\lambda\mathrm{Id}, \lambda\mathrm{Id}) \in Z(V)$ which is exactly that of scalar multiplication from k. This suggests that $Z(V)$ knows about the additional scalars that may exist for V.

Definition 1.14. A Jordan Pair V is called central if every $a \in Z(V)$ is of the form $a = (\lambda\mathrm{Id}, \lambda\mathrm{Id})$ for some $\lambda \in k$

Central Jordan Pairs are Jordan Pairs where k already contains all the scalars or actions on V that preserve the structure.

Although $Z(v)$ can bring in new scalars or new actions on V the resulting algebra may no longer be a Jordan Pair. $Z(V)$ is often not commutative or even a subalgebra of $\mathrm{End}(V^+) \times \mathrm{End}(V^-)$, meaning $Z(V)$ may not be a scalar extension of k.

Proposition 1.15. *Let V be a Jordan Pair with $Z(V)$ a commutative subalgebra of $End(V^+) \times End(V^-)$. Then V can be a Jordan Pair over the extension $Z(V)$.*

Furthermore the structure of $Z(V)$ depends on the structure of V, specifically the ideals of V.

Proposition 1.16. *Let V be a Jordan Pair. If V is simple then $Z(V)$ is an extension field of k.*

1.4 Pseudo-Inverses

And application to differential geometry I assume? or some other flavor of geometry

Conclusion

I will call it there but like anything there is an endless amount of things to learn

References

[1] The Book of Involutions.

[2] *A Taste of Jordan Algebras*. Universitext. Springer-Verlag, New York, 2004.

[3] Ottmar Loos. *Jordan Pairs*, volume 460 of *Lecture Notes in Mathematics*. Springer, Berlin, Heidelberg, 1975.