Reviewing the Subobj. classifier

What is a Subobject (Subset, Subtype)

spec. → Injective functions (monomorphisms)

Replacement

$$\begin{tikzcd} A \arrow[rr, leftrightarrow] \arrow[dr, hook] & & A' \arrow[dl, hook] \\ & B & \end{tikzcd}$$

Really a family of "Equivalent" injective functions

Replacement → characteristic functions $\chi : B \to \{T, F\}$

spec. The "subobject" is then $\{b \in B \mid \chi(b) = T\}$

In Sets spec. $\{x \in A \mid P(x) = true\}$ ← preimages

replacement: $\{My : y \in B\}$ ← images

When are these Equal?

$A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$

Topos and Subobj classifiers give use a framework for this that don't rely on specific axioms

Which is the right approach? obviously characteristic functions! monomorphism loose specificity (only have "up to iso") and with a subobject classifier we can get the relevent monomorphisms and subobjects as a pull back

last time

Given $\Omega = \{True, False\}$ and $B \xrightarrow{\chi} \Omega$

$$\begin{tikzcd} & 1 \arrow[d, "true"] \\ B \arrow[r, "\chi"] & \Omega \end{tikzcd} \qquad \begin{tikzcd} * \arrow[d] \\ True \end{tikzcd}$$

There existed an A s.t

$$\begin{tikzcd} A \arrow[r, "\exists!"] \arrow[d, hook, "\iota"'] & 1 \arrow[d, "true"] \\ B \arrow[r, "\chi"] & \Omega \end{tikzcd}$$

And we could make it universal!

$$\begin{tikzcd} \forall S \arrow[rrd, bend left, "\exists!"] \arrow[ddr, bend right, "g"'] & & \\ & A \arrow[r, "\exists!"] \arrow[d, hook, "\iota"'] & 1 \arrow[d, "true"] \\ & B \arrow[r, "\chi"] & \Omega \end{tikzcd} \qquad \begin{tikzcd} S \arrow[rrd, bend left, "!"] \arrow[dr, dashed, "\exists! u"] \arrow[ddr, bend right, "g"'] & & \\ & A \arrow[r, "!"] \arrow[d, hook, "\iota"'] & 1 \arrow[d, "true"] \\ & B \arrow[r, "\chi"] & \Omega \end{tikzcd}$$

For χ we can create a subobject $A \overset{\iota}{\hookrightarrow} B$ s.t that any other subset S that satisfys χ factors through $A \hookrightarrow B$

But what is a Subobject classifier? Another universal thing

is a map $true : 1 \to \Omega$ quantifying

$$\begin{tikzcd} 1 \arrow[d, "true"] \\ \Omega \end{tikzcd}$$

Then for any monomorphism or really a family of them

$$\begin{tikzcd} A \arrow[r, "\exists!"] \arrow[d, hook, "\iota"'] & 1 \arrow[d, "true"] \\ B & \Omega \end{tikzcd}$$

Then there is a unique characteristic function for ι: making it a pull back

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook, "\iota"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "true"] \\ B \arrow[r, "\chi"] & \Omega \end{tikzcd}$$

So maybe monomorphism was the "right approach"? as we get χ uniquely

Maybe this question ~~misses~~ misses the point and actually monomorphisms and char. function is a chicken and egg situation!

So I guess monomorphisms came first

Punch line: family of monomorphisms/~ into B $\overset{\text{1-to-1}}{\underset{\text{correspondence}}{\longleftrightarrow}}$ morphisms $B \to \Omega$

$$Sub(B) \cong \Omega^B \leftarrow Hom(B, \Omega) \leftarrow \text{classifying or characteristic functions}$$

A foundation for The Subobject
→ A UMP that can be built on
and Improved. Can also be Typed but for now we will skip

Topos: A place to do math. Is a category with
- products
- terminal objects
- exponentials (Ex functions have types)

} Cartisian closed category
a place you can curry
and co-curry

- Has limits for finite diagrams
  - Ex products $A \times B$ is limit for diagram $A \quad B$
  - Pullback is limit for diagram $A \to B \leftarrow C$
- Has Subobject classifier $\left(\text{Subob} : \left\{ \begin{array}{c} A \\ \downarrow \iota \\ B \end{array} \right\} = Sub(B) \cong \Omega^B = \{ B \to \Omega \} \right)$

Claim: You can do math in topos (including logic)
Claim: Type is a topos

Example of doing math in the topos Type
Define Ring a product of types.
$(R, +, -, 0_R, \cdot, 1_R)$: Ring where $+ : R \times R \to R$
Define module over R similarly
$(M, +, -, 0_M, *)$: ${}_R\text{Mod}$

propositions: M is simple : $\forall N \subseteq M \quad N = 0$ or $N = M$
as a Type / Prop $(N : Sub(M) \times (N \text{ is module})) \longmapsto (N = 0 \sqcup N = M)$