

# Reviewing the Subobj. Classifier

What is a Subobject (Subset, Subtype)

→ Injective functions (monomorphisms)  $A \hookrightarrow A'$

Really a family of "Equivalent" injective functions

→ Characteristic functions  $\chi: B \rightarrow \{T, F\}$

The "Subobject" is then  $\{b \in B \mid \chi(b) = T\}$

In Sets  $\text{Spec. } \{x \in A \mid P(x) = \text{true}\}$   
 Replacement:  $\{M \mid y \in B\}$   
 When are these Equal?  $\leftarrow$  Images

$$A = B \Leftrightarrow A \subseteq B \text{ and } B \subseteq A$$

Topos and Subobj classifiers give use a framework for this that don't rely on specific axioms.

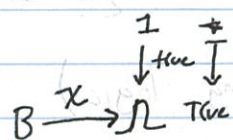
Which is the right approach? Obviously characteristic functions!

Monomorphism A loose Specificity (only have "up to iso") and with a Subobject classifier

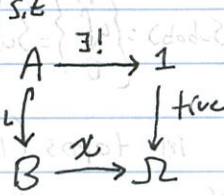
we can get the relevant monomorphisms and Subobjects as a pull back

Given  $\Omega = \{\text{True}, \text{False}\}$

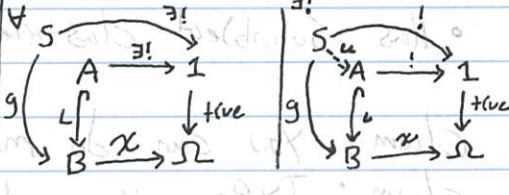
and  $B \xrightarrow{\chi} \Omega$



There existed an A



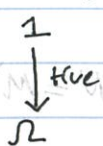
And we could make it universal!



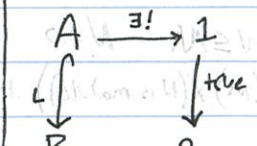
For  $\chi$  we can create a Subobject  $A \hookrightarrow B$  s.t that any other subset S that satisfies  $\chi$  factors through  $A \hookrightarrow B$

But what is a Subobject classifier? Another universal thing

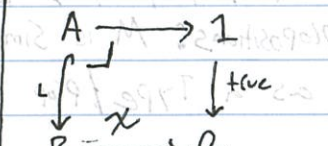
is a map  $\text{true}: 1 \rightarrow \Omega$  quantifying



Then for any monomorphism or really a family of them



Then there is a unique characteristic function for L: making it a pull back



So maybe monomorphism was the "right approach"? as we get  $\chi$  uniquely  
 Maybe this question misses the point and actually monomorphisms and char. function is a chicken and egg situation!

So I guess monomorphisms came first

Punch line: Family of monomorphisms/n into B  $\xleftrightarrow{\text{correspondence}}$  morphisms  $B \rightarrow \Omega$

$$\text{Sub}(B) \cong \Omega^B \leftarrow \text{Hom}(B, \Omega) \leftarrow \text{classifying or characteristic functions}$$

## A foundation for The Subobject

→ A UMP that can be built on

$A \geq B$  and Improved. Can also be Typed but for now we will skip

Topos: A place to do math, is a Category with

- products
  - terminal objects
  - exponentials (Ex functions <sup>have</sup> ~~two~~ types)
  - Has ~~the~~ limits for finite diagrams
- Cartesian closed category  
a place you can Curry  
and Co-Curry

Ex products  $A \times B$  is limit for diagram  $A \quad B$

Pullback is limit for diagram  $A \rightarrow B \leftarrow C$

- Has Subobject classifier  $(\text{Sub}(B) = \{ \downarrow \begin{matrix} A \\ B \end{matrix} \} = \text{Sub}(B) \cong \Omega^B = \{ B \rightarrow \Omega \})$

Claim: You can do math in topos (including logic)

Claim: Type is a topos

Example of doing math in the topos Type

Define Ring a product of types.

$(R, +, -, \cdot, 0_R, 1_R)$ : Ring where  $+ : R \times R \rightarrow R$

Define module over R similarly

$(M, +, -, \cdot, 0_M, *)$ :  $R$  Mod

Propositions:  $M$  is Simple:  $\forall N \subseteq M \quad N=0$  or  $N=M$

as a Type / Prop  $(N: \text{Sub}(M) \times (N \text{ is module})) \mapsto (N=0 \sqcup N=M)$