Switching Equivalence of Systems of Lines over Finite Fields

Ian Jorquera
Joint with: Emily J. King

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Colorado State University

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All the information about equiangular lines over finite fields, which are frames, is encoded in the triple products.

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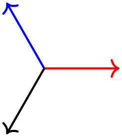
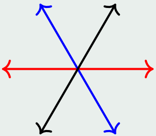
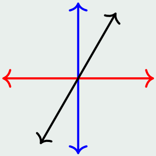
Optimal Line Packings Over $\mathbb{R}$ and $\mathbb{C}$



Line Packings: Pack n lines in $\mathbb{R}^d$ or $\mathbb{C}^d$, where every line is maximally spread apart.





Goal: Maximize pairwise interior angles, or minimize $\cos^2\theta$

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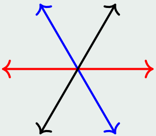

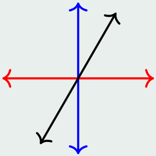

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Given n lines, represent each by a unit vector

$$\Phi = \begin{bmatrix} | & | & & | \\ \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ | & | & & | \end{bmatrix} \in \mathbb{F}^{d \times n}$$

New Goal: Minimize $\max_{i \neq j} | \langle \varphi_i, \varphi_j \rangle |^2$

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How To Find Grassmannian Frames:

Step 1: Find a lower bound on coherence.

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How To Find Grassmannian Frames:

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Step 2: Find examples which meet the bound.

$\mu^2(\Phi) \geq 0 \Rightarrow \Phi = (\varphi_j)_{j=1}^n$ orthonormal is a Grassmannian frame.

The Welch-Rankin Bound and Equiangular Tight Frames.

For $\Phi = \left[\varphi_1, \ldots, \varphi_n\right]$ in $\mathbb{F}^d$.

Welch-Rankin Bound (Welch; 1974) (Rankin; 1955)

$$\mu^2(\Phi) = \max_{i \neq j} | \langle \varphi_i, \varphi_j \rangle |^2 \geq \frac{n-d}{d(n-1)}$$

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With equality if and only if

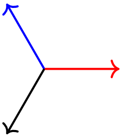
- **Equiangular**: $|\langle \varphi_i, \varphi_j \rangle|^2 = b$ for all $i \neq j$
- **Tightness**: $\Phi\Phi^* = cI$

} Φ is an ETF

Tightness generalize the Pythagorean theorem or Parceval's identity

$$\Phi\Phi^* = cI \Leftrightarrow \sum_{i=1}^{d} \|\langle x, \varphi_i \rangle \varphi_i\|^2 = c\|x\|^2$$

“Optimal” Line Packings over Finite Fields



Discretizing Reality: Finite Field Analog to $\mathbb{R}^d$

Real IP Spaces	$\rightsquigarrow$	Orthogonal Geometries
$\mathbb{R}^d$	$\rightsquigarrow$	$\mathbb{F}_q^d$, where $q = p^\ell$ is odd.
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Example: Non-degeneracy as a proof of being non-zero

$V = \mathbb{F}_3^3$ with $\langle x, y \rangle = x^\intercal y$ the dot product.

$$x = \begin{bmatrix}1\\1\\1\end{bmatrix} \quad \left\langle \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\rangle = 0 \quad \left\langle \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix} \right\rangle = 1$$

Discretizing the Imaginary: Finite Field Analog to $\mathbb{C}^d$

Complex IP Spaces	$\rightsquigarrow$	**Unitary Geometries**
$\mathbb{C}^d$	$\rightsquigarrow$	$\mathbb{F}_{q^2}^d$, where $q = p^\ell$.
$x \mapsto \overline{x}$	$\rightsquigarrow$	$x \mapsto x^q$
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$\langle x, y \rangle = \overline{\langle y, x \rangle}$	$\rightsquigarrow$	$\langle x, y \rangle = \langle y, x \rangle^q$
$\langle x, - \rangle : \mathbb{C}^d \to \mathbb{C}$ linear		$\langle x, - \rangle : \mathbb{F}_{q^2}^d \to \mathbb{F}_{q^2}$ linear
$\langle x, x \rangle > 0$ iff $x \neq 0$	$\rightsquigarrow$	$\langle x, y \rangle \neq 0$ for some $y \in \mathbb{F}_{q^2}^d$ iff $x \neq 0$

Geometric Algebra (not algebraic geometry)

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Case O ($\mathbb{F} = \mathbb{F}_q$)

$V = \mathbb{F}_q^d$ (choosing a basis)
$\langle x, y \rangle = x^\mathsf{T} M y$, where $M^\mathsf{T} = M$ and is invertible.

There is a basis for V such that $M = \mathrm{Diag}(1, \ldots, 1, \delta)$

- $\delta = 1$ is a square
- δ is not a square

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The **adjoint** of A, is the map $A^\dagger$ such that $\langle Ax, y \rangle = \left\langle x, A^\dagger y \right\rangle$.

A map $A : V \to V$ is called a **unitary** if $\langle Ax, Ay \rangle = \langle x, y \rangle$ $(A^\dagger A = I)$

Types of Geometry over $\mathbb{R}$ and $\mathbb{C}$

The Real Case

$V = \mathbb{R}^d$
$\langle x, y \rangle = x^\intercal M y$, where $M^\intercal = M$ and is invertible.

There is a basis for V such that $M = \text{Diag}(1, \ldots, 1, -1, \ldots, -1)$

If $M = \text{Diag}(1, \ldots, 1)$, then $\langle x, y \rangle$ is an Inner product.

The Complex Case

$V = \mathbb{C}^d$
$\langle x, y \rangle = x^* M y$, where $M^* = M$ and is invertible.

There is a basis for V such that $M = \text{Diag}(1, \ldots, 1)$, and $\langle x, y \rangle$ is an inner product.

For inner products: $A^\dagger = A^*$.

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Inner Product Spaces:

- Subspaces of inner product spaces are inner product spaces
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Case O and U: Not the case. Consider an orthogonal geometry $V = \mathbb{F}_3^4$ with $\langle x, y \rangle = x^\intercal y$

$$\Phi = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 2 & 2 \\ 0 & 1 \end{bmatrix} \quad \Phi^\dagger \Phi = [\langle \varphi_i, \varphi_j \rangle] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\mathrm{im}\,\Phi \subseteq V$ is degenerate.

Frame Theory (Greaves, Iverson, Jasper, Mixon; 2022), (J, King; 2025)

Let $\Phi = \left[\varphi_1, \varphi_2 \ldots, \varphi_n\right]$ from $V = \mathbb{F}^d$, $a, b, c \in \mathbb{F}$. Then Φ is a

- **frame** for $\operatorname{im}\Phi$ if $\operatorname{im}\Phi$ is non-degenerate $\Leftrightarrow \mathrm{rk}(\Phi) = \mathrm{rk}(\Phi^\dagger\Phi)$
- c-**tight frame** for $\operatorname{im}\Phi$ if $\Phi\Phi^\dagger = cI$ on $\operatorname{im}\Phi$
- (a, b)-**equiangular** if
  - $\langle\varphi_j, \varphi_j\rangle = a$ for all j
  - $\langle\varphi_j, \varphi_k\rangle\langle\varphi_k, \varphi_j\rangle = b$ for all $j \neq k$
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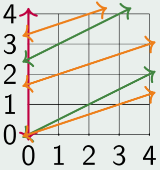
Example: $V = \mathbb{F}_5^2$ with $\langle x, y\rangle = x^\intercal My$, where $M = \mathrm{Diag}(1, 3)$

$$\Phi = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 1 & 1 \end{bmatrix} \qquad \Phi^\dagger\Phi = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$



Φ is a $(2, 1, 3)$-ETF for $\mathbb{F}_5^2$ of $n = 3$ vectors.

Frame Theory: ETFs in case O and U

$V = \mathbb{F}_3^4$ with $\langle x, y \rangle = x^\intercal M y$, where $M = \mathrm{Diag}(1,1,1,2)$

$$\Phi = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 2 & 2 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 2 & 1 & 2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Φ is a $(0,1,0)$-ETF for $\mathbb{F}_3^4$ of $n = 10$ vectors.

$V = \mathbb{F}_{3^2}^5$ with $\langle x, y \rangle = x^* y$. Ψ is a $(0,1,0)$-ETF of 16 vectors.

$$\Psi = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & a & a & a & a & a^3 & a^3 & a^3 & a^3 \\ a & a & a^5 & a^5 & a^5 & a^5 & a^5 & a^5 & 1 & 1 & 1 & 1 & a^6 & a^6 & a^6 & a^6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & a & a & a^5 & a^5 & a^3 & a^3 & a^7 & a^7 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & a & a^5 & a & a^5 & a^3 & a^7 & a^3 & a^7 \\ 0 & 0 & a^2 & a^6 & 0 & 0 & 0 & 0 & a^7 & a^3 & a^3 & a^7 & a & a^5 & a^5 & a \end{bmatrix}$$

Implications for Reality

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If Φ is a ETF of n vectors for $\mathbb{C}^d$ then there exists ETFs of n vectors in $\mathbb{F}_{q^2}^d$, in Case U, for infinity many fields with distinct characteristics.

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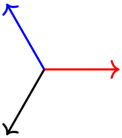
A similar theorem is known for Case O, and the converse is known!

Theorem for Case O (Greaves, Iverson, Jasper, Mixon; 2022)

If Φ is an ETF of n vectors for $\mathbb{F}_q^d$, in case O, with $\text{char}\mathbb{F}_q > 2n - 5$ then there exists a real ETF of n vectors for $\mathbb{R}^d$

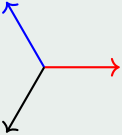
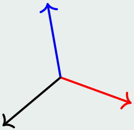
Equivalence of Line Packings



When are systems of lines the "same"? (J, King; 2025)

Let $\Phi = (\varphi_j)_{j=1}^n$ and $\Psi = (\psi_j)_{j=1}^n$ be collections of lines in $\mathbb{F}^d$.

Unitary Equivalence: if $\Psi = U\Phi$, $U \in U(\mathbb{F}^d)$. $\Rightarrow$ $\Psi^\dagger \Psi = \Phi^\dagger \Phi$

$\Phi =$  $\Psi =$ 

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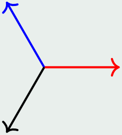
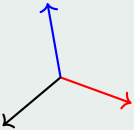
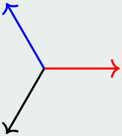
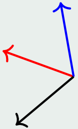
$\Phi =$  $\Psi =$ 

Switching Equivalence: if $\Psi = U\Phi T$ where $T = \text{diag}(t_1, \ldots, t_n)$, and $|d_i|^2 = 1$ $\Rightarrow$ $\Psi^\dagger \Psi = T^\dagger \Phi^\dagger \Phi T$

$\Phi =$  $\Psi =$ 

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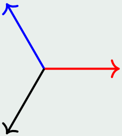
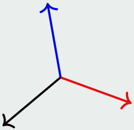
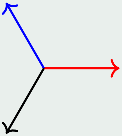
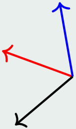
$\Phi =$  $\Psi =$ 

Switching Equivalence: if $\Psi = U\Phi T$ where $T = \text{diag}(t_1, \ldots, t_n)$, and $|d_i|^2 = 1$ $\Leftrightarrow$ $\Psi^\dagger \Psi = T^\dagger \Phi^\dagger \Phi T$

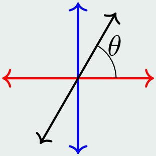
$\Phi =$  $\Psi =$ 

Invariants of Switching Equivalence

Let $\Phi = (\varphi_j)_{j=1}^n$ be a frame for $\mathbb{F}^d$.

m-product (m-vertex Bargmann invariant)

Double Product: $\Delta(\varphi_j, \varphi_k) = \langle \varphi_j, \varphi_k \rangle \langle \varphi_k, \varphi_j \rangle$



In $\mathbb{R}$ or $\mathbb{C}$ this is "measuring" the geodesic distance. $\Delta(\varphi_j, \varphi_k) = \cos^2(\theta)$

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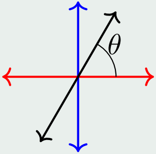
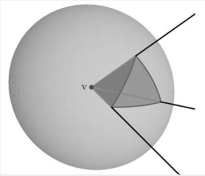


Triple Product: $\Delta(\varphi_j, \varphi_k, \varphi_\ell) = \langle \varphi_j, \varphi_k \rangle \langle \varphi_k, \varphi_\ell \rangle \langle \varphi_\ell, \varphi_j \rangle$

In $\mathbb{R}$ or $\mathbb{C}$ this is "measuring" the symplectic area of a geodesic triangle.



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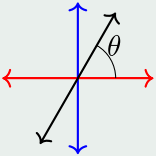

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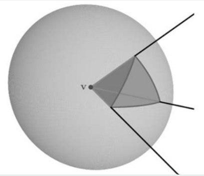

m-**Product:**

$$\Delta(\varphi_{j_1}, \varphi_{j_2}, \dots, \varphi_{j_m}) = \langle \varphi_{j_1}, \varphi_{j_2} \rangle \langle \varphi_{j_2}, \varphi_{j_3} \rangle \cdots \left\langle \varphi_{j_{m-1}}, \varphi_{j_m} \right\rangle \langle \varphi_{j_m}, \varphi_{j_1} \rangle$$

Invariants of Equivalence

Let $\Phi = (\varphi_j)_{j=1}^n$ and $\Psi = (\psi_j)_{j=1}^n$ be frames for $\mathbb{F}^d$.

m-products are invariants

If Φ and Ψ switching equivalent $(\Psi = U\Phi T)$ then all m-products are equal $(1 \leq m \leq n)$.

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m-products are invariants

If Φ and Ψ switching equivalent ($\Psi = U\Phi T$) then all m-products are equal ($1 \leq m \leq n$).

Theorem (Gallagher, Proulx; 1977), (Chien, Waldron; 2016), (J, King; 2025)

If $\Delta(\varphi_{j_1}, \varphi_{j_2}, \ldots, \varphi_{j_m}) = \Delta(\psi_{j_1}, \psi_{j_2}, \ldots, \psi_{j_m})$ for all $1 \leq m \leq n$ then Φ and Ψ switching equivalent

In general, you need to look at all the m-products!

But sometimes 3 is plenty!

Let $\Phi = (\varphi_j)_{j=1}^n$ and $\Psi = (\psi_j)_{j=1}^n$ be frames for $\mathbb{F}^d$ ($\mathbb{F}$ is finite in case O or U).

Theorem (J, King; 2025)

If for all $j \neq k$ we have that $\langle \varphi_j, \varphi_k \rangle \neq 0$ and $\langle \psi_j, \psi_k \rangle \neq 0$ then Φ and Ψ are switching equivalent if and only if all double and triple products agree.

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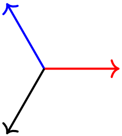
For this reason triple products have been widely used to study the algebraic properties of frames. See (Appleby et al.; 2011), (Zhu; 2015), and (King; 2019).

The Welch Bound Revisited



A Flashback: Understanding ETFs in $\mathbb{R}$ and $\mathbb{C}$

For $\Phi = [\varphi_1, \dots, \varphi_n]$ in $\mathbb{F}^d$.

Welch-Rankin Bound (Welch; 1974) (Rankin; 1955)

$$\mu^2(\Phi) = \max_{i \neq j} |\langle \varphi_i, \varphi_j \rangle|^2 \geq \frac{n-d}{d(n-1)}$$

With equality if and only if

- **Equiangular**: $|\langle \varphi_i, \varphi_j \rangle|^2 = b$ for all $i \neq j$
- **Tightness**: $\Phi\Phi^* = cI$

$\Bigg\}$ Φ is an ETF

A Flashback: Understanding ETFs in $\mathbb{R}$ and $\mathbb{C}$

ETFs (optimal line packings) are understood in two ways

Geometrically as ETFs

- **Equiangular**: $i \neq j$ $|\langle \varphi_i, \varphi_j \rangle|^2 = b$
- **Tightness**: $\Phi\Phi^* = cI$

Combinatorially with

$$b = \frac{n-d}{d(n-1)}$$

- $n = \#$ lines
- $d =$ dimension

Do we get this over Finite Fields?

Short answer: No.

Long answer: Sort of!

On the Failure of a Welch-Rankin Equality

Theorem (Greaves, Iverson, Jasper, Mixon; 2022)

If Φ is a (a, b, c)-ETF for $V = \mathbb{F}^d$ then $d(n-1)b \equiv (n-d)a^2$
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Example: $V = \mathbb{F}_5^7$ with $\langle x, y \rangle = x^\intercal y$

$$\Phi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 & 4 & 0 & 2 \\ 0 & 1 & 1 & 2 & 1 & 2 & 2 & 3 \\ 1 & 0 & 1 & 2 & 3 & 2 & 2 & 3 \\ 1 & 1 & 0 & 2 & 3 & 4 & 4 & 1 \end{bmatrix}$$

Φ is a $(2, 1)$-equiangular frame for V.
It satisfies $b \equiv 1 \equiv \frac{1}{49} 2^2 \equiv \frac{n-d}{d(n-1)} a^2$.

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It satisfies $b \equiv 1 \equiv \frac{1}{49}2^2 \equiv \frac{n-d}{d(n-1)}a^2$. But Φ is not a tight frame

A New Hope: Using Sums of Triple Products

Let $\mathbb{F}$ be a field with $\mathrm{char}\mathbb{F} \nmid dn$, and $V = \mathbb{F}^d$.
Let $\Phi = [\varphi_1, \ldots, \varphi_n]$ for V be an (a, b)-equiangular frame for V $(a \neq 0)$.

Theorem (J, King; 2025)

Φ is an $(a, b, na/d)$-ETF if and only if

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- $d(n-1)b = (n-d)a^2$
- $\displaystyle\sum_{\ell=1}^{n} \Delta(\varphi_j, \varphi_k, \varphi_\ell) = \frac{nab}{d}$ for all $j \neq k$

Applications of this Welch-Rankin Equality

Theorem (J, King; 2025)

Let $\Phi = \left[\varphi_1, \ldots, \varphi_n\right]$ be an (a, b)-equiangular frame for $\mathbb{F}_q^d$ in Case O. If

- $\text{char}\mathbb{F} \nmid d(d+1)$, and $a \neq 0$, $a^2 \neq b$
- $\Gamma \subseteq \Phi$ of $d+1$ vectors, where $\Delta(\varphi_j, \varphi_k, \varphi_\ell)$ is constant for all distinct $\varphi_j, \varphi_k, \varphi_\ell \in \Gamma$

Then Γ is a regular d-simplex, an $(a, b, \frac{(d+1)a}{d})$-ETF

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More Results (J, King; 2025)

- Can be further generalized to sub-sets of Φ, in either case O or U, being k-simplices, but has an additional condition on sums of triple products.

Simplices

$V = \mathbb{F}_3^4$ with $\langle x, y \rangle = x^\intercal M y$, where $M = \mathrm{Diag}(1,1,1,2)$

$$\Phi = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 2 & 2 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 2 & 1 & 2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Φ is a $(0,1,0)$-ETF for $\mathbb{F}_3^4$ of $n = 10$ vectors.

- Φ is a maximal ETF for $\mathbb{F}_3^4$
- No 4×10 real ETF is known to exist
- Contains 30 regular 3-simplices: 15 square geometry, 15 non-square geometry, both pairs of 15 form $(10, 4, 2)$-BIBDs

Questions

$$\Phi = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

A $(0,1,1)$-ETF for $\mathbb{F}_3^3$